Development of the thermal boundary layer at the inlet of a circular pipe and comparison with Lévêque solution

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This paper addresses the problem of laminar flow with fully developed velocity and with developing temperature in a circular pipe with either uniform wall temperature or uniform wall heat flux. Invoking the method of lines, the applicable energy equation accounting for constant properties has been discretized in the axial direction only. As a result of this, the transformed boundary value problem represented by a single ordinary differential equation where the radius is the independent variable was reduced to a system of algebraic equations and solved numerically. This simple technique applied to the case of uniform wall temperature is parallel to that used by Lévêque, and the comparisons between both approaches are in good quantitative agreement. Predictions for the extended case of uniform wall heat flux are in accord with the exact solutions as well. Multiple discretizations of the energy equation in the axial direction are directly related to the first and the second order terms in the so-called Lévêque series using perturbation analysis. Both procedures yield fairly accurate results when compared with the solutions provided by the truncated Lévêque series. The proposed methodology exhibits a peculiar analogy with the conduction heat transfer mechanism along a disk fin placed perpendicular to the fluid stream.

Keywords: Lévêque problem; method of lines; disk fin

Introduction

The situation of a hydrodynamically developed and thermally developing flow in an isothermal circular pipe constitutes a historical problem in the theory of laminar forced convection. In 1928, Lévêque[1] attempted to solve this problem by using a similarity transformation technique based on the flat plate concept. His idea relied on the hypothesis that the thermal boundary layer was thin compared to the hydrodynamic boundary layer inside the pipe. Correspondingly, Lévêque's method is valid in the thermal entrance region of the pipe very close to the origin of the heat exchange section.

Some extensions of the Lévêque solution has been reported by Mercer,[2] Worsøe-Schmidt,[3] Newman,[4] Richardson,[5] and Shih and Tsou.[6] These authors solved the governing energy equation by a perturbation analysis. This procedure gave rise to the so-called Lévêque series in the convective heat transfer literature. Mercer obtained the first two terms analytically and the next two terms numerically. Worsøe-Schmidt presented in closed form up to the first-order term and higher-order terms up to $n=6$ numerically. Newman found an analytical solution up to second-order terms and Richardson extended his analytical procedure up to third-order terms. Shih and Tsou were able to extend Newman's solution up to fourth-order terms utilizing numerical techniques. In addition to these investigations, recent studies on this subject have been reported by Huang *et al.*[7] and by Gottifredi and Flores.[8] Both papers resort to rather cumbersome mathematical procedures that necessitate the numerical evaluation of intricate functions.

In light of the foregoing discussion, the objective of the present work is to provide an approximate, but very simple solution for the problem of thermal entrance exposed to uniform wall temperature and its ramifications in the region of validity envisioned by the original work of Lévêque. Another related problem that deserves to be examined here is the situation involving uniform wall heat flux. To accomplish these goals, the applicable energy equation is first transformed into an ordinary differential equation by discretizing the axial derivative within the framework of the method of lines.[9] The resulting boundary value problem governs the temperature profile of the fluid flow along a line ($0 \leqslant r \leqslant R$) placed at a certain axial station downstream in the pipe. Its solution may be obtained either in *closed form* or *numerically*.

One peculiarity of the transformed boundary value problem is that it is analogous to the formulation describing steady-state conduction along a disk fin having the wall boundary conditions imposed at its extreme.

Furthermore, possible refinements of this procedure are also discussed, giving rise to the utilization of two and three consecutive lines in the downstream part of the pipe. Thus, it can be expected that the approximate solution will improve as the number of lines increases. In a sense, these extensions are comparable with the calculations of first- and second-order terms in the Lévêque series using perturbation analysis, but without necessitating the evaluation of intricate functions.

Prediction of the mean bulk temperature distribution for case Ⓣ shows good agreement with the original Lévêque solution and its generalized series up to second-order terms. Further comparisons involving higher-order terms are straightforward, but were not performed here. Additionally, calculations of the wall temperature distribution were presented for case Ⓗ. Even with only one line, numerical results show trends which are in accord with the exact solution for the entire region of thermal development.

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Problem statement

Under the assumption of constant properties and fully developed laminar velocity, the growth of the thermal boundary layer in a circular pipe is characterized by the following set of equations.

Case Ⓣ:

$$(1-\eta^2)\frac{\partial\theta}{\partial X}=\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\theta}{\partial\eta} \quad (1)$$

$$\theta=0, \qquad X=0 \quad (2)$$

$$\frac{\partial\theta}{\partial\eta}=0, \qquad \eta=0 \quad (3)$$

$$\theta=1, \qquad \eta=1 \quad (4)$$

where the dimensionless temperature, θ, is given by

$$\theta=(T-T_w)/(T_e-T_w) \quad (5)$$

Case Ⓗ

$$(1-\eta^2)\frac{\partial\phi}{\partial X}=\frac{\partial^2\phi}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial\phi}{\partial\eta} \quad (6)$$

$$\phi=0, \qquad X=0 \quad (7)$$

$$\frac{\partial\phi}{\partial\eta}=0, \qquad \eta=0 \quad (8)$$

$$\frac{\partial\phi}{\partial\eta}=1, \qquad \eta=1 \quad (9)$$

where the dimensionless temperature is now

$$\phi=k(T-T_e)/q_wR \quad (10)$$

The thermal quantities of interest in the above-mentioned problems include the *mean bulk temperature distribution*

$$\theta_b \text{ (or } \phi_b)=4\int_0^1 \theta \text{ (or } \phi)(1-\eta^2)\eta\, d\eta \quad (11)$$

For Case Ⓣ:

the mean Nusselt number distribution

$$\overline{Nu}=-\ln\theta_b/2X \quad (12)$$

For Case Ⓗ:

the wall temperature distribution

$$\phi_w=\phi(1,X) \quad (13)$$

In addition, the local Nusselt number distribution is another quantity that is usually reported in problems of this nature.

Numerical methodology

Equations 1 and 6 will be systematically reformulated, invoking a hybrid method of solution called the method of lines.[9] In general terms, this method seeks to replace a partial differential equation in two independent variables by an appropriate system of ordinary differential equations in one of these variables. Specifically, for a partial differential equation of parabolic type like Equations 1 and 6, a direct application of the method requires the replacement of the radial derivatives by suitable finite-difference analogs, while retaining the axial derivatives as continuous. Correspondingly, depending on the number of radial intervals selected, this simple procedure usually leads to a system of ordinary differential equations of first-order subject to the entrance boundary conditions.

Conversely, instead of adopting the above-mentioned traditional approach, we have explored a variant of the method of lines in this paper. In this alternative approach, the axial derivatives in Equations 1 and 6 have been expressed in finite-difference form, while the radial derivatives are maintained in continuous form. Accordingly, this rather crude approach yielded a differential-difference equation for each case examined, which is valid at a line $(0\leqslant r\leqslant R)$ drawn at a fixed axial station, say X_1, along the pipe. Hence, this transformed boundary value problem describes approximately the thermal behavior of the fluid flow along this particular line in the cross-section of the pipe. In other words, this simple approach allows for an approximate calculation of the development of the thermal boundary layer at a certain axial station, X_1, inside the pipe.

One station (one line)

Case Ⓣ:

Using a backwards finite-difference analog for the axial derivative in Equation 1 results in the following differential-difference

Notation

D	Pipe diameter
h	Convection coefficient
k	Thermal conductivity
Nu	Nusselt number, hD/k
Pe	Peclet number, $\bar{u}D/\alpha$
q_w	Wall heat flux
r	Radial coordinate
R	Pipe radius
T	Temperature
$\bar{U}$	Mean velocity
x	Axial coordinate
X	Dimensionless axial coordinate, x/RPe

Greek letters

α	Thermal diffusivity
η	Dimensionless radial coordinate
θ	Dimensionless temperature for case Ⓣ, $(T-T_w)/(T_e-T_w)$
ϕ	Dimensionless temperature for case Ⓗ, $k(T-T_e)/q_wR$

Subscripts

b	Mean bulk
e	Entrance
Ⓗ	Wall heat flux
Ⓣ	Wall temperature
w	Wall
1	First station
2	Second station
3	Third station

equation

$$\frac{d^2\theta_1}{d\eta^2}+\frac{1}{\eta}\frac{d\theta_1}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\theta_1=0 \tag{14}$$

This equation, which incorporates the entrance boundary condition of Equation 2 already, is valid at an axial station $X=X_1$ (see Figure 1(a)) and is subject to the radial boundary conditions

$$\frac{d\theta_1}{d\eta}=0, \quad \eta=0 \tag{15}$$

$$\theta_1=1, \quad \eta=1 \tag{16}$$

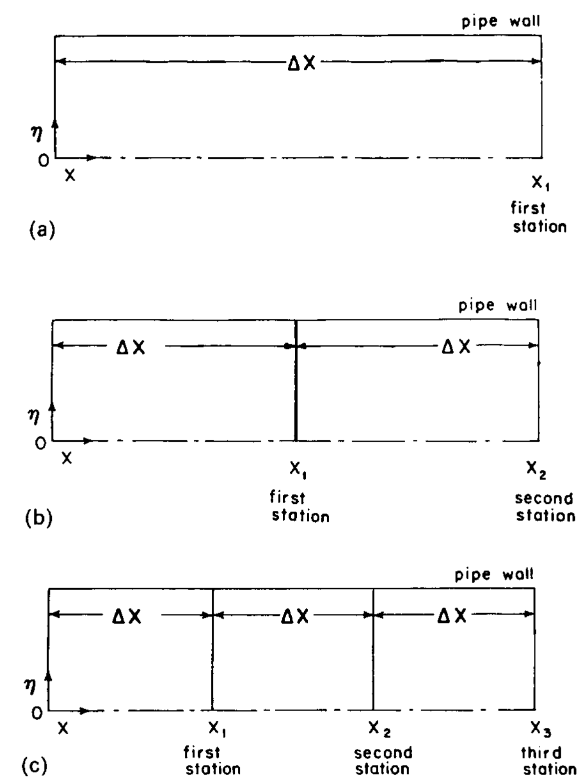

Figure 1

Case Ⓗ:

Similarly, for this situation Equation 6 may be applied to station X_1 (see Figure 1(a)), resulting in an equation identical to Equation 14, but in terms of ϕ. This becomes

$$\frac{d^2\phi_1}{d\eta_2}+\frac{1}{\eta}\frac{d\phi_1}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\phi_1=0 \tag{17}$$

The relevant boundary conditions are given by

$$\frac{d\phi_1}{d\eta}=0, \quad \eta=0 \tag{18}$$

$$\frac{d\phi_1}{d\eta}=1, \quad \eta=1 \tag{19}$$

Results

Actually, the set of Equations 14–16 and 17–19 may be solved analytically or numerically. Details of the numerical calculations may be obtained by the subroutine PASVA3[10] utilizing 20 uniform intervals. These solutions provide the temperature profiles at any axial station X_1 in the thermal development region. In turn, these profiles combined with Equations 11 and 13 supply the mean bulk temperature and the mean Nusselt number respectively at each particular station X_1.

Computed results of θ_b for Ⓣ are presented in Table 1 under the heading stating 1 line. A comparison between these predictions and the results of the Graetz series recalculated by Shah[11] with 121 terms yield fairly accurate agreement. The error at $X=10^{-5}$ is only 0.09%. Far downstream at $X=10^{-2}$, the error is 2.27%, while the corresponding error given by Lévêque solution is of the order of 3%. At this juncture, it should be mentioned that it is traditionally stated that the classical Lévêque solution gives excellent predictions for the thermal entrance region where $0<X<10^{-2}$.[4,12]

Additionally, values for the wall temperature for Ⓗ calculated with 1 line are illustrated in Table 2. Here again, the basis for comparison purposes is the series solution reevaluated by Shah[11] retaining 121 terms. At $X=10^{-4}$, the discrepancy is 15.48% and it is drastically reduced as X increases reaching a value of 4.29% at 1.4×10^{-1}. Nevertheless, although the errors may look quite large, the absolute deviations at these two locations are 0.01 and 0.04 units respectively.

Table 1 Comparison of mean bulk temperature for Ⓣ

0.00		Lévêque series [4]						Method of lines							
X	Graetz series [11]	1 term	%e	2 terms	%e	3 terms	%e	1 line	%e	2 lines	%e	2 lines*	%e	3 lines	%e
10^{-5}	0.99814					0.99814		0.99902	0.09	0.99901	0.09				
6×10^{-5}	0.99391					0.99391		0.99525	0.13	0.99502	0.11	0.99663	0.27	0.99493	0.10
10^{-4}	0.99147					0.99147		0.99295	0.15	0.99255	0.11				
6×10^{-4}	0.97251							0.97568	0.33	0.97431	0.19	0.98190	0.97	0.97381	0.13
10^{-3}	0.96175	0.95930	0.25	0.96170	0	0.96175	0	0.96605	0.45	0.96417	0.25	0.97487	1.36		
6×10^{-3}	0.88060	0.86462	1.81	0.88002	0.07	0.88050	0.01	0.89408	1.53	0.88795	0.83	0.92472	5.01	0.88570	0.58
10^{-2}	0.83622	0.81110	3	0.83510	0.13	0.83606	0.02	0.85521	2.27	0.84652	1.23	0.89899	7.51	0.84330	0.85
1.2×10^{-2}	0.81690							0.83844	2.64	0.82857	1.43	0.88813	8.72	0.82494	0.98

* based on a backwards formulation with three points

Table 2 Comparison of wall temperature for Ⓗ

X	Graetz series [11]	Method of lines											
		1 line	%e	Δ	2 lines	%e	Δ	2 lines*	%e	Δ	3 lines	%e	Δ
10^{-4}	0.05835	0.06738	15.48	0.00903	0.06822	16.91	0.00987	0.06577	12.72	0.00742	0.06854	17.46	0.01019
10^{-3}	0.13048	0.12362	−5.26	−0.00686	0.12916	1.01	−0.00132	0.12803	−1.87	−0.00245	0.13123	0.57	0.00075
10^{-2}	0.30689	0.27921	−9.02	−0.02768	0.29386	4.25	−0.01333	0.29381	−4.26	−0.01308	0.29905	−2.55	−0.00784
5×10^{-2}	0.60000	0.55359	−7.74	−0.04641	0.57790	3.68	−0.02210	0.57705	−3.83	−0.02295	0.58652	−2.25	−0.01348
10^{-1}	0.84308	0.79405	−5.82	−0.04903	0.81934	2.82	−0.02374	0.81500	−3.33	−0.02808	0.82800	−1.79	−0.01500
1.4×10^{-1}	1.01288	0.96946	−4.29	−0.04342	0.99343	1.92	−0.01945	0.98500	−2.75	−0.02788	1.00119	−1.15	−0.01169

* based on a backwards formulation with three points

Further considerations

Although the computed results based on a one line approximation are encouraging and indeed compatible with those furnished by Lévêque solution in its region of validity ($X \leqslant 10^{-2}$), it seems convenient to explore some natural refinements that do not necessitate the use of special functions. In order to accomplish this goal, first the axial coordinate X will be divided in two equal intervals, ΔX, defining two consecutive lines (see Figure 1(b)). Furthermore, the situation involving three equal intervals ΔX giving three consecutive lines will be also examined (see Figure 1(c)). At this point, it should be said that the idea behind this is to have an identical number of differential equations as that given by the extensions of the Lévêque solution using perturbation analysis.

Two consecutive stations (two lines)

Case Ⓣ:

Rewriting Equation 1 at station $X_2 = 2\Delta X$ (see Figure 1(b)), the following differential-difference equation is obtained

$$\frac{d^2\theta_2}{d\eta^2}+\frac{1}{\eta}\frac{d\theta_2}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\theta_2+\frac{1}{\Delta X}(1-\eta^2)\theta_1=0 \tag{20}$$

employing the backwards formulation with two points for the axial derivative. In the preceding equation, θ_1 designates the temperature profile at the axial station X_1 in the forcing function. The applicable boundary conditions are given by

$$\frac{d\theta_2}{d\eta}=0, \qquad \eta=0 \tag{21}$$

$$\theta_2=1, \qquad \eta=1 \tag{22}$$

Case Ⓗ:

In light of the foregoing statement, the descriptive set of equations for this case is readily written as

$$\frac{d^2\phi_2}{d\eta^2}+\frac{1}{\eta}\frac{d\phi_2}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\phi_2+\frac{1}{\Delta X}(1-\eta^2)\phi_1=0 \tag{23}$$

$$\frac{d\phi_2}{d\eta}=0, \qquad \eta=0 \tag{24}$$

$$\frac{d\phi_2}{d\eta}=1, \qquad \eta=1 \tag{25}$$

Three consecutive stations (three lines)

Case Ⓣ:

The governing equation (1) specialized at station $X_3 = 3\Delta X$ (see Figure 1(c)) becomes

$$\frac{d^2\theta_3}{d\eta^2}+\frac{1}{\eta}\frac{d\theta_3}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\theta_3+\frac{1}{\Delta X}(1-\eta^2)\theta_2=0 \tag{26}$$

where the axial derivative has been replaced by a backwards formulation having two points. Here, θ_2 corresponds to the temperature profile associated to station X_2. The relevant boundary conditions are

$$\frac{d\theta_3}{d\eta}=0, \qquad \eta=0 \tag{27}$$

$$\theta_3=1, \qquad \eta=1 \tag{28}$$

Case Ⓗ:

Utilizing the same procedure discussed previously, Equation 26 is rephrased in terms of ϕ giving

$$\frac{d^2\phi_3}{d\eta^2}+\frac{1}{\eta}\frac{d\phi_3}{d\eta}-\frac{1}{\Delta X}(1-\eta^2)\phi_3+\frac{1}{\Delta X}(1-\eta^2)\phi_2=0 \tag{29}$$

which is subject to the appropriate boundary conditions

$$\frac{d\phi_3}{d\eta}=0, \qquad \eta=0 \tag{30}$$

$$\frac{d\phi_3}{d\eta}=1, \qquad \eta=1 \tag{31}$$

Results

The numerical solution of Equations 14–16 and 17–19 was based on standard finite-difference techniques, giving rise to a system of algebraic equations. As before, the number of preselected intervals for the analysis of two and three consecutive stations was 20. A simple computer program was prepared in order to handle these calculations in a systematic manner and the numerical values are tabulated in Table 1 for case Ⓣ and in Table 2 for case Ⓗ, respectively.

Case Ⓣ:

The error distribution utilizing two lines decreases gradually when compared with those using one line at the same stations. In particular, at $X=10^{-2}$ the error is 1.23% using two lines. At the same axial position, the Lévêque series retaining two terms produces an error of 0.13%. Similarly, the same tendency is manifested for the formulation accounting for three lines. Employing three lines, the calculated error at $X=10^{-2}$ is 0.98%, while a three-term Lévêque series has a discrepancy of 0.02%.

An anomalous situation is observed when two lines are used in conjunction with a backwards formulation accounting for three points. It can be observed that the errors in the thermal development region are much larger than those related to a backwards formulation involving two points.

Case Ⓗ:

Using a grid consisting of two lines, the errors associated with the wall temperature are drastically reduced, except at $X=10^{-4}$, where it increased slightly. For the entire region tested $0<X<1.4\times10^{-1}$, the average absolute deviation is in the vicinity of 0.015 units only. The local deviations show a decreasing behavior in the downstream direction. For the same grid having two lines, an additional calculation was carried out utilizing a backwards formulation with three points. As can be observed, the errors of this approximation are larger than those based on a backwards formulation involving two points. The last situation analyzed corresponds to the approximation using three lines. As excepted, the distribution of errors is drastically reduced when compared with those given by the formulation of two lines, with the exception of $X=10^{-4}$. At $X=10^{-1}$, the relative error is -1.79%. For the region examined, the average absolute deviation for the wall temperature is 0.01 units.

Steady-state conductive analogy

One station

A detailed examination of Equation 14 reveals its resemblance with the equation controlling a steady-state conduction process in a radial fin.[13] In particular, we are dealing here with a disk fin placed *normal* to a fluid stream inside a pipe and insulated on the other side. A uniform dimensionless temperature having a value $\theta=1$ is imposed at its periphery for case Ⓣ. In addition to this, the equivalent convective coefficient is considered proportional to the laminar velocity profile, being a maximum at the centerline and dropping to zero at the extreme. The effective thermal conductivity of the disk fin is inversely proportional to the magnitude of the selected axial station X_1 in the downstream region.

Alternatively, the same concept applies for the case Ⓗ, but with the consideration that the disk fin has now a uniform peripherical heat flux, whose value in dimensionless form is $d\phi_1/d\eta=1$.

Two consecutive stations

The disk fin analogy explained for the situation involving one station still prevails here, but with the addition of a heat generation term that depends on X and η.

Three consecutive stations

Similarly, the steady-state process of the disk fin at station X_3 incorporates a heat generation term affected by X and η.

Conclusions

One of the main findings that has emerged from the presentation of results is that the method of lines discretizing the axial derivative with one step gives essentially the same results than those invoking the Lévêque solution. In a mathematical context, this methodology leads to a system of algebraic equations at a particular axial station in the thermal development region, which may be solved in a straightforward manner. On the other hand, Lévêque solution necessitates the use of special functions whose numerical evaluation is quite elaborate. The same contrasting patterns are observed when both procedures are extended to situations involving multiple stations with the purpose of improving the numerical results. As expected, accuracy must improve significantly as the number of axial stations is increased.

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